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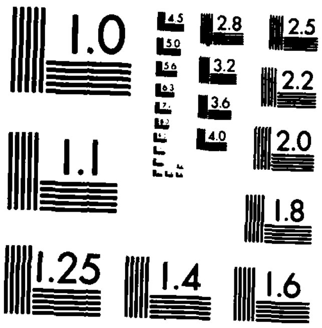
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ON EXISTENCE AND UNIQUENESS OF SOLUTIONS OF HAMILTON-JACOBI EQUATIONS

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ABSTRACT

The theory of scalar first order nonlinear partial differential equations has been enjoying a rapid development in the last few years. This development occurred because the authors established uniqueness criteria for generalized solutions - called viscosity solutions - which correctly identify the solutions sought in areas of application, including control theory, differential games and the calculus of variations. The concept of viscosity solutions is relatively easy to work with and many formally heuristic or difficult proofs have been made rigorous or simple using this concept. A feedback process has begun and the experience recently gained in working with viscosity solutions has suggested new existence and uniqueness results. The current paper continues this interaction by establishing new existence and uniqueness results in a natural generality suggested by earlier proofs. It is also felt that the presentation of the comparison results, which imply uniqueness, continuous dependence, and are used to estimate moduli of continuity, has something to offer over earlier presentations in special cases.

AMS (MOS) Subject Classifications: 35F20, 35F25, 35L60

Key words: Hamilton - Jacobi equations, first order nonlinear partial differential equations, existence, comparison theorems, viscosity solutions, moduli of continuity.

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ON EXISTENCE AND UNIQUENESS OF SOLUTIONS OF HAMILTON - JACOBI EQUATIONS

Michael G. Crandall and Pierre-Louis Lions*

Introduction.

Our main purpose here is to establish some uniqueness and existence theorems for Hamilton-Jacobi equations. We will focus on the Cauchy problem

(CP) $$u_t + H(x,t,u,Du) = 0 \text{ in } \mathbb{R}^N\times(0,T),\ u(x,0) = \varphi(x),$$

and the stationary problem

(SP) $$u + H(x,u,Du) = 0 \quad \text{in } \mathbb{R}^N,$$

in which H is a real-valued function of its arguments, x denotes points in $\mathbb{R}^N$, Du stands for the spatial gradient $(u_{x_1},\dots\dots,u_{x_N})$ of u, which is itself a real-valued function of either (x,t) or x as appropriate.

The uniqueness follows from comparison theorems of maximum principle type. The comparison theorems we present here correspond to some results proved in special cases by Ishii in [9] and used by him without proof in more general cases in [8]. These results concern the comparison of solutions which are uniformly continuous in x - but perhaps unbounded - in cases where the Hamiltonian H is restricted by appropriate continuity conditions, but not otherwise by growth conditions. We will give complete proofs in a natural generality which strictly includes the corresponding statements of [8]. In any case, we would feel it worthwhile to present our complete proofs even in the special cases used in [9]. For other sorts of uniqueness results for unbounded solutions see Ishii [9] and Crandall and Lions [4], [5].

As regards existence, we will prove existence results in the same new generality for which we establish the uniqueness. The main tool used for this is estimates on the modulus of continuity of solutions of (CP) and (SP). These estimates are obtained by using the comparison results and modifications of the

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elegant ideas of Ishii [8].

The notion of "solution" of the differential equations in (CP) and (SP) that we will employ is that of a "viscosity solution". We will not explain this notion here and refer the reader instead to Crandall, Evans and Lions [2] for easy access to definitions and some proofs of the sort presented here in the simplest situations and to Crandall and Lions [3], where various equivalent notions of viscosity solutions were presented and the first uniqueness proofs were given. The question of existence of solutions of Hamilton - Jacobi equations has a long history. We refer the reader to the book Lions [10] for references to the substantial work which predates the notion of viscosity solutions. Existence theorems in the general spirit of this paper were established in model cases in Crandall and Lions [3]. Considerable generality was achieved in the results of Lions [10], [11] (in which boundary problems were considered - we do not discuss existence under boundary conditions here, although it is obvious that one could), some new results (in $\mathbb{R}^N$) were obtained by Souganidis [12] and then arguments were introduced in Barles [1] which brought the existence theory to a new level of generality. Ishii [8] (using estimates of moduli) and Crandall and Lions [4] (relying partly on Barles' methods) were the first works concerning the existence of (possibly) unbounded viscosity solutions. The review paper [7] outlines other aspects of the theory of viscosity solutions and contains a moderately current bibliography.

The text begins with a Section 1 which is devoted to preliminaries and statements of the main existence and comparison theorems. The comparison results are then proved in Section 2. Section 3 is devoted to studying the moduli of continuity of solutions of (CP) and (SP). The results on the moduli of continuity quickly imply the desired existence theorems.

We are grateful to H. Ishii for pointing out a defect in a previous draft of this paper.

Section 1. Preliminaries, Notation and Statements of Results

We will use the following notational conventions throughout. The set of continuous functions which map a metric space Ω into the reals will be denoted by $C(\Omega)$ and the subspace of $C(\Omega)$ consisting of uniformly continuous functions will be denoted by $UC(\Omega)$.

Functions of modulus of continuity type will be denoted by the letter m as well as m with various subscripts (e.g., m_R, m_H, etc.). Such functions m map $[0,\infty)$ into $[0,\infty)$ and satisfy:

$$
(1.1)\qquad
\begin{array}{ll}
\text{(i)} & m \text{ is nondecreasing,} \\
\text{(ii)} & m(0+) = 0, \\
\text{(iii)} & m(a + b) \le m(a) + m(b) \text{ for } a,\, b \ge 0, \\
\text{(iv)} & m(r) \le m(1)(r + 1) \text{ for } r \ge 0,
\end{array}
$$

where (iv) is in fact a consequence of (i) and (iii). We will call any such function "a modulus".

When Ω is a subset of $\mathbb{R}^N$, $UC_x(\Omega\times[0,T])$ denotes the space of those $u \in C(\Omega\times[0,T])$ for which there is a modulus m and an $r > 0$ such that

$$|u(x,t) - u(y,t)| \le m(|x - y|) \text{ for } x,y \in \Omega,\ |x - y| \le r \text{ and } t \in [0,T].$$

Finally, in what follows, $T > 0$ and $B_R(z)$ denotes the closed R-ball centered at $z \in \mathbb{R}^N$, $B_R(z) = \{\, x \in \mathbb{R}^N\colon |x - z| \le R\}$, while $\operatorname{int}B_R(z)$ is its interior. The Hamiltonians we will consider will typically be restricted by the following set of conditions:

(H0) $\quad H \in C(\mathbb{R}^N\times[0,T]\times\mathbb{R}\times\mathbb{R}^N)$.

(H1) $\quad$ For each $R > 0$ there is a modulus m_R such that for all $(x,t,p),\ (y,t,q) \in \mathbb{R}^N\times[0,T]\times B_R(0)$

$$|H(x,t,r,p) - H(y,t,r,q)| \le m_R(|p - q| + |x - y|).$$

(H2) $\quad H(x,t,r,p)$ is nondecreasing in r for all (x,t,p) in $\mathbb{R}^N\times[0,T]\times\mathbb{R}^N$.

There is a modulus m_H such that for all $(t,r) \in [0,T]\times\mathbb{R}$

(H3) and all $x, y \in \mathbb{R}^N$ and $\lambda > 0$

$$H(x,t,r,\lambda(x-y)) - H(y,t,r,\lambda(x-y)) \geq -m_H(\lambda|x-y|^2 + |x-y|).$$

Remark 1. The assumption (H3) is a replacement for the stronger requirement

There is a modulus m_H such that for all $(t,r,p) \in [0,T]\times\mathbb{R}\times\mathbb{R}^N$ and

(H3)' all $x, y \in \mathbb{R}^N$

$$|H(x,t,r,p) - H(y,t,r,p)| \leq m_H(|x-y|(1+|p|)),$$

which was introduced in [3] and has been used since then. The significant observation that the original proofs of uniqueness go through under (H3) (although one should choose φ to be radial in, e.g., the proof of [3, Theorem II.1], as is done later in [3]) is due to R. Jensen who was led to this remark by considering the uniqueness question for Hamilton - Jacobi equations in infinite dimensions in an ongoing investigation. It is easy to see that (H0)-(H3) is strictly weaker than (H0), (H1), (H2), (H3)'. For example, if $N = 1$, $H(x,t,p) = b(x)p$ satisfies the weaker set of conditions if and only if b is bounded and uniformly continuous and $b(x) + cx$ is nondecreasing for some c, while the stronger conditions replace this last requirement with Lipschitz continuity of b. The analogous remark holds for general N. (Later we will comment on how (H3) itself may be significantly relaxed.) Finally, the requirement that m_H be a modulus may seem restrictive, but it is not - if this estimate holds with any nondecreasing function tending to 0 at 0+, then it holds with a modulus.

The uniqueness and existence results for (CP) are:

Theorem 1. Let H satisfy (H0) - (H3). Then:

Comparison. Let Ω be an open subset of $\mathbb{R}^N$, $\bar{\Omega}$ its closure and $\partial\Omega$ its boundary. Let and $u, v \in UC_x(\bar{\Omega}\times[0,T])$ satisfy the differential inequalities

(1.1) $\quad u_t + H(x,t,u,Du) \leq 0$ and $0 \leq v_t + H(x,t,v,Dv)$ on $\Omega\times(0,T)$

in the viscosity sense. Assume also that

(1.2) $\qquad u(x,0) \le v(x,0)$ on Ω and $u(x,t) \le v(x,t)$ on $\partial\Omega\times[0,T]$.

Then $u \le v$ on $\Omega\times[0,T]$.

<u>Existence</u>. Let $\varphi \in UC(\mathbb{R}^N)$. Then (CP) has a solution $u \in UC_x(\mathbb{R}^N\times[0,T])$ in the viscosity sense.

<u>Remark 2</u>. Theorem 1 extends, via the substitution $v = e^{-\lambda t}u$, to Hamiltonians H for which $H(x,t,u,p) + \lambda u$ satisfies the assumptions in place of H for some λ.

Of course, the "comparison" theorem when formulated in this way gives uniqueness and much more - it also implies continuous dependence and will be used to estimate moduli of continuity, etc.. The parallel theorem for (SP) results upon interpreting (H0) - (H3) for a function $H:\mathbb{R}^N\times\mathbb{R}\times\mathbb{R}^N \to \mathbb{R}$ by regarding H as a (t-independent) special case of the functions considered above. We have:

<u>Theorem 2</u>. Let $H \in C(\mathbb{R}^N\times\mathbb{R}\times\mathbb{R}^N)$ satisfy (H0) - (H3). Then:

<u>Comparison</u>. Let Ω be an open subset of $\mathbb{R}^N$ and $u, v \in UC(\bar{\Omega})$ satisfy the differential inequalities

(1.3) $\qquad u + H(x,u,Du) \le 0$ and $0 \le v + H(x,v,Dv)$ on Ω

in the viscosity sense and $u \le v$ on $\partial\Omega$. Then $u \le v$ on Ω.

<u>Existence</u>. (SP) has solution $u \in UC(\mathbb{R}^N)$.

<u>Remark 3</u>. Assume that $\partial\Omega$ is of class C^2 and ν_x denotes the inward normal to Ω at $x \in \partial\Omega$. Then the assumed comparison $u \le v$ of $\partial\Omega\times[0,T]$ in Theorem 1 and on $\partial\Omega$ in Theorem 2 can be weakened by replacing $\partial\Omega$ by the set of $x \in \partial\Omega$ for which $\lambda \to H(x,t,u,p + \lambda\nu_x)$ is not nonincreasing for some $(t,u,p) \in [0,T]\times\mathbb{R}\times\mathbb{R}^N$ and variants of Theorems 1 and 2 remain true. One needs to strenghten (H3) near the boundary a bit - see Crandall and Newcomb [6].

The next lemma collects some simple estimates used throughout the text for convenient reference. It may be skipped at this time.

Lemma 1. Let m be a modulus. Then:

(i) If ε, δ, $r > 0$, $m(1) > 0$, $\delta + m(r) \geq r^2/\varepsilon$ and ε and δ are sufficiently small, then

$$r \leq 2(m(1)\varepsilon)^{1/2} \text{ and } r^2/\varepsilon \leq \delta + m(2(m(1)\varepsilon)^{1/2}). \tag{1.4}$$

(ii) If $E \geq 0$, $F > m(1)$ and $K(\gamma)$ is defined for $\gamma > 0$ by

$$K(\gamma) = \sup\{\, m(\gamma Er + (r/F)^{1/\gamma}) - r \colon 0 \leq r \leq F\},$$

then $K(0+) = \lim_{\gamma \downarrow 0} K(\gamma) = 0$.

Proof. We first sketch the proof of (i). The assumptions and (1.1)(iv) yield

$$r^2 \leq \varepsilon(m(1)(1 + r) + \delta) \leq 4\varepsilon \max(\delta, m(1), m(1)r)$$

and, choosing the various possibilities for the max on the right, the worst estimate on r which arises for small δ, ε is the one claimed in (1.4). Then the second estitmate arises from the monotonicity of m. (Obviously these estimates can be improved, but we will not need to do so.)

We just sketch the simple proof of the (awkward looking) assertion (ii). Let r_γ be the maximum point of

$$f_\gamma(r) = m(\gamma Er + (r/F)^{1/\gamma}) - r$$

on $[0,F]$. Clearly $f_\gamma(r_\gamma) \geq m(0) - 0 = 0$. If $(r_\gamma/F)^{1/\gamma}$ has a limit point θ as $\gamma \downarrow 0$ and $0 < \theta$, then the corresponding limit superior of $f_\gamma(r_\gamma)$ is at most $m(1) - F < 0$ (so this cannot be), while if $\theta = 0$ then the limit superior is at most $m(0) - 0 = 0$. Since $K(\gamma) = f_\gamma(r_\gamma)$, we are done.

We conclude this section with a useful lemma which illustrates the advantage one can take of the linear dependence on the time derivative in (CP). This is a convenient packaging of a standard argument. (For a sketch of a proof in a special case see Ishii [8, Lemma 3.3]).

Lemma 2. Let Ω be an open subset of $\mathbb{R}^N$, $H, G \in C(\Omega\times(0,T)\times\mathbb{R}\times\mathbb{R}^N)$ and $u, v \in C(\Omega\times(0,T))$ satisfy

$$u_t + H(x,t,u,Du) \leq 0 \text{ and } 0 \leq v_t + G(x,t,v,Dv) \text{ on } \Omega\times(0,T).$$

Define $z(x,y,t)$ on $\Omega\times\Omega\times(0,T)$ by $z(x,y,t) = u(x,t) - v(y,t)$. Then

$$z_t + H(x,t,u(x,t),D_x z) - G(y,t,v(y,t),-D_y z) \leq 0 \text{ on } \Omega\times\Omega\times(0,T)$$

in the viscosity sense.

<u>Proof</u>. It will suffice to show that if $\varphi \in C^1(\Omega\times\Omega\times(0,T))$ and

$$(1.5) \qquad (x,y,t) \to u(x,t) - v(y,t) - \varphi(x,y,t)$$

has a strict local maximum at $(a,b,c) \in \Omega\times\Omega\times(0,T)$, then

$$(1.6) \qquad \varphi_t(a,b,c)+H(a,c,u(a,c),D_x\varphi(a,b,c))-G(b,c,v(b,c),-D_y\varphi(a,b,c)) \leq 0.$$

Put

$$(1.7) \qquad d(x,y,t,s) = \max(|x - a|,|y - b|,|t - c|,|s - c|).$$

Since the maximum is strict, there is an $r_0 > 0$ and $H \in C([0,r_0])$ with $H(r) > 0$ for $0 < r < r_0$ such that

$$(1.8) \qquad u(x,t)-v(y,t)-\varphi(x,y,t) \leq u(a,c)-v(b,c)-\varphi(a,b,c)-H(d(x,y,t,t))$$

for $0 \leq d(x,y,t,t) \leq r_0$.

We claim: Given $r > 0$ there is an $\varepsilon > 0$ such that

$$(1.9) \qquad \Psi(x,y,t,s) = u(x,t) - v(y,s) - \varphi(x,y,t) -(t - s)^2/\varepsilon$$

has a local maximum point (x_r,y_r,t_r,s_r) which satisfies

$$(1.10) \qquad d(x_r,y_r,t_r,s_r) \leq r.$$

Assuming the claim is correct for the moment we then have, by the assumptions,

$$\frac{2(t_r - s_r)}{\varepsilon} + \varphi_t(x_r,y_r,t_r) + H(x_r,t_r,u(x_r,t_r),D_x\varphi(x_r,y_r,t_r)) \leq 0 ,$$

$$-\frac{2(t_r - s_r)}{\varepsilon} - G(y_r,t_r,v(y_r,t_r),- D_y\varphi(x_r,y_r,t_r)) \leq 0.$$

If these inequalities are added, r is sent to 0 and (1.10) is taken into account, the result follows.

It remains to produce (x_r,y_r,t_r,s_r). Fix $r < r_0$ and choose ε so that

$$(1.11) \quad |v(y,t)-v(y,s)|+|u(x,t)-u(x,s)|+|\varphi(x,y,t)-\varphi(x,y,s)| \leq (t-s)^2/\varepsilon+H(r)/2.$$

for $d(x,y,t,s) \le r_0$. This is clearly possible. Let Ψ be given by (1.9) and $d(x,y,t,s) = r$. Then either $d(x,y,t,t) = r$ or $d(x,y,s,s) = r$. Assume $d(x,y,t,t) = r$ (the other case is treated a similar way). Then using (1.8) and (1.11) we have

$$\Psi(x,y,t,s) \le u(x,t) - v(y,t) - \varphi(x,y,t) - (v(y,s) - v(y,t) + (t-s)^2/\varepsilon) \le$$
$$\le \Psi(a,b,c,c) - H(r) + |v(y,s) - v(y,t)| - (t-s)^2/\varepsilon \le$$
$$\le \Psi(a,b,c,c) - H(r)/2,$$

so the maximum of Ψ over the set $d \le r$ must occur at an interior point, which is then a local maximum satisfying (1.10).

Section 2. Proofs of the Comparison Results.

We will first prove Theorem 1, which is more difficult than Theorem 2. The next lemma is an essential step in the proof of the comparison assertion of Theorem 1.

Lemma 3. Let the assumptions of the comparison assertion of Theorem 1 hold. Then $u - v$ is bounded from above on $\Omega \times [0,T]$.

Proof of Lemma 3. Since $u, v \in UC_x(\Omega \times [0,T])$, there is a modulus m and an $r > 0$ such that

$$|u(x,t) - u(y,t)| + |v(x,t) - v(y,t)| \le m(|x-y|) \quad \text{for } x, y \in \bar{\Omega} \text{ and } |x-y| \le r. \tag{2.1}$$

Let

$$d(x) = \text{distance}(x, \partial\Omega). \tag{2.2}$$

Since $u \le v$ on $\partial\Omega \times [0,T]$ and (2.1) holds, we have

$$u(x,t) - v(y,t) \le m(|x-y|) + m(\min(d(x),d(y))) \text{ for } x, y \in \bar{\Omega} \text{ and } t \in [0,T] \text{ such that } |x-y| \le r \text{ and } \min(d(x),d(y)) \le r. \tag{2.3}$$

Thus to bound $u - v$ it will suffice to bound $u(a,t) - v(a,t)$ for $a \in \Omega$ and $d(a) \ge 2r$. To this end, choose such an a and set

$$\begin{aligned} &K = 1 + 2m(r)/r^2, \\ &\Psi(x,y,t) = u(x,t) - v(y,t) - K(|x-a|^2 + |y-a|^2), \\ &w(t) = \sup\{\Psi(x,y,t) : (x,y) \in B_r((a,a))\}, \end{aligned} \tag{2.4}$$

for $0 \le t \le T$. Here

$$B_r((a,a)) = \{(x,y) \in \mathbb{R}^N \times \mathbb{R}^N : |x - a|^2 + |y - a|^2 \le r^2\} \subset \bar{\Omega} \times \bar{\Omega}$$

by $d(a) > r$. In order to prove the lemma, it suffices to show that w is bounded from above uniformly in a. Clealy w is continuous. The main part of the proof is to show that w' is bounded above on the open set

$$\{w > 0\} = \{t \in [0,T]: w(t) > 0\}$$

by a constant in the viscosity sense (and hence in the distribution sense - see [2, Proposition I.11]). Since $u \le v$ on $\Omega \times \{0\}$, we have

$$w(0) \le \sup\{u(x,0) - v(y,0) - K(|x - a|^2 + |y - a|^2) : (x,y) \in B_r((a,a))\} \le 2m(r).$$

If also $w' \le C$ on $\{w > 0\}$, we conclude that $w(t) \le 2m(r) + CT$ on $[0,T]$, whence the result. It remains to show that w' is bounded above on $\{w > 0\}$. The constant K in (2.4) is choosen so as guarantee that $\Psi(x,y,t) < \Psi(a,a,t)$ if $(x,y) \in \partial B_r((a,a))$, so there is a point $(\bar{x},\bar{y}) \in \operatorname{int} B_r((a,a))$ such that

(2.5) $$w(t) = u(\bar{x},t) - v(\bar{y},t) - K(|\bar{x} - a|^2 - |\bar{y} - a|^2).$$

Of course, $\bar{x}$, $\bar{y}$ depend on t and a. Let I be an open interval in $\{w > 0\}$, $\rho \in C^1(I)$ and $w(t) - \rho(t)$ have a local maximum at $\bar{t} \in I$. Let $(\bar{x},\bar{y}) \in \operatorname{int} B_r((a,a))$ be such that (2.5) holds with $t = \bar{t}$. Then for all (x,y,t) near $(\bar{x},\bar{y},\bar{t})$

$$u(x,t) - v(y,t) - K(|x-a|^2 + |y-a|^2) - \rho(t) \le u(\bar{x},\bar{t}) - v(\bar{y},\bar{t}) - K(|\bar{x}-a|^2 - |\bar{y}-a|^2) - \rho(\bar{t}).$$

It then follows from Theorem 1, Lemma 2 and the definition of viscosity solutions that

(2.6) $$\rho'(\bar{t}) + H(\bar{x},\bar{t},u(\bar{x},\bar{t}),2K(\bar{x} - a)) - H(\bar{y},\bar{t},v(\bar{y},\bar{t}),-2K(\bar{y} - a)) \le 0.$$

We will use this with the estimates given next to reach our conclusion. Clearly

(2.7) $$2K|\bar{x} - a|,\ 2K|\bar{y} - a| \le 2Kr.$$

The string of inequalities written next will be explained immediately below.

$$\begin{aligned} &H(\bar{x},\bar{t},u(\bar{x},\bar{t}),2K(\bar{x} - \bar{a})) - H(\bar{y},\bar{t},v(\bar{y},\bar{t}),-2K(\bar{y} - a)) \ge \\ (2.8)\quad &\ge H(\bar{x},\bar{t},v(\bar{y},\bar{t}),2K(\bar{x}-\bar{a})) - H(\bar{y},\bar{t},v(\bar{y},\bar{t}),-2K(\bar{y} - a)) \ge \\ &\ge -m_{2Kr}(|\bar{x} - \bar{y}| + 2K|\bar{x} - a| + 2K|\bar{y} - a|) \\ &\ge -m_{2Kr}(2r(1 + 2K)). \end{aligned}$$

The first inequality is due to the assumption $w(\bar{t}) > 0$ (which implies $u(\bar{x},\bar{t}) > v(\bar{y},\bar{t}) > 0$) and (H2). In the next inequality, we use (2.7) and (H1). The final inequality uses $|\bar{x} - \bar{y}| < 2r$ and (2.7). From (2.6) and (2.8) we now know that that, in the viscosity sense,

$$w' < C = m_{2Kr}(2r(1 + 2K)) \text{ on } \{ w > 0 \}.$$

and the proof of Lemma 3 is complete.

Remark 4. The reader may observe that only (H0) - (H2) were used in the proof of Lemma 3 and (H3) was not required. An analogous remark holds concerning the study of (SP) later.

End of Proof of Comparison for (CP).

Assume the hypotheses of the theorem, that (2.1) holds and that $u < v$ fails. We may choose a constant c such that

(2.9) $\quad c > 0$ and $M = \sup\{u(x,t) - v(x,t) - ct\colon x \in \bar{\Omega} \text{ and } 0 < t < T\} > 0.$

Note that $M < \infty$ by Lemma 3. Next let ε, $\beta > 0$ and set

(2.10) $\quad \Psi(x,y,t) = u(x,t) - v(y,t) - ct - (|x - y|^2/\varepsilon + \beta(|x|^2 + |y|^2)).$

Clearly

(2.11) $$\delta(\beta) = M - \sup\{\Psi(x,x,t)\colon x \in \bar{\Omega},\ 0 < t < T\}$$

satisfies

(2.12) $$\delta(0+) = 0$$

and

(2.13) $\quad \sup\{\Psi(x,y,t)\colon (x,y,t) \in \bar{\Omega}\times\bar{\Omega}\times[0,T] \text{ and } |x - y| < r\} > M - \delta(\beta).$

In view of (2.1), for $(x,y,t) \in \bar{\Omega}\times\bar{\Omega}\times[0,T]$ satisfying $|x - y| < r$ we have

$$M + m(|x - y|) - (|x - y|^2/\varepsilon + \beta(|x|^2 + |y|^2)) >$$

(2.14) $$> u(x,t) - v(x,t) - ct +(v(x,t) - v(y,t)) -(|x - y|^2/\varepsilon +$$

$$+ \beta(|x|^2 + |y|^2)) = \Psi(x,y,t)$$

and from (2.13) and (2.14) it is clear that Ψ attains a positive maximum with respect to the set $\{(x,y,t) \in \bar{\Omega}\times\bar{\Omega}\times[0,T]\colon |x - y| < r \}$ at some point $(\bar{x},\bar{y},\bar{t})$ if β is small (as we hereafter assume).

We now proceed to make some simple and standard estimates involving $(\bar{x},\bar{y},\bar{t})$. Using (2.14) and (2.13) we have

$$\delta(\beta) + m(|\bar{x} - \bar{y}|) \geq |\bar{x} - \bar{y}|^2/\varepsilon + \beta(|\bar{x}|^2 + |\bar{y}|^2). \tag{2.15}$$

From (2.15) and Lemma 1 we deduce that (taking $m(1) \neq 0$ without loss of generality)

$$\begin{aligned} &\text{(i)} && |\bar{x} - \bar{y}| \leq 2(m(1)\varepsilon)^{1/2}, \\ \text{(2.16)}\quad &\text{(ii)} && |\bar{x} - \bar{y}|^2/\varepsilon \leq \delta(\beta) + m(2(m(1)\varepsilon)^{1/2}), \\ &\text{(iii)} && \beta(|\bar{x}|^2 + |\bar{y}|^2) \leq \delta(\beta) + m(2(m(1)\varepsilon)^{1/2}), \end{aligned}$$

provided only that ε and β are small. Next we check that $\bar{t} > 0$. Indeed, the inequality

$$\begin{aligned} \Psi(x,y,0) &= u(x,0)-v(x,0)+v(x,0)-v(y,0)-(|x-y|^2/\varepsilon+\beta(|x|^2+|y|^2)) \\ &\leq m(|x - y|) - (|x - y|^2/\varepsilon + \beta(|x|^2 + |y|^2)) \end{aligned}$$

coupled with (2.16) shows that

$$\Psi(\bar{x},\bar{y},0) \leq m(2(m(1)\varepsilon)^{1/2})$$

so $\Psi(\bar{x},\bar{y},0) \geq M - \delta(\beta)$ is impossible if ε and β are small. Thus $\bar{t} > 0$. Similarly, if $d(\bar{x}) \leq r$ or $d(\bar{y}) \leq r$, (2.3) yields

$$\Psi(\bar{x},\bar{y},\bar{t}) \leq 2m(r),$$

which cannot be if r is sufficiently small (and we may take it to be so). Lastly, (2.16) (i) guarantees that $|\bar{x} - \bar{y}| < r$ if ε is sufficiently small. Thus we can guarantee that $(\bar{x},\bar{y},\bar{t})$ is an interior point of the set

$$\{(x,y,t) \in \Omega \times \Omega \times [0,T]: |\bar{x} - \bar{y}| \leq r \}$$

for which it provided a maximum of Ψ.

To proceed, put

$$p = 2(\bar{x} - \bar{y})/\varepsilon, \quad q_x = 2\beta\bar{x}, \quad q_y = -2\beta\bar{y}. \tag{2.17}$$

Since Ψ has an interior maximum point at $(\bar{x},\bar{y},\bar{t})$, Lemma 2 and the definition of viscosity solution yield

$$c + H(\bar{x},\bar{t},u(\bar{x},\bar{t})),p + q_x) - H(\bar{y},\bar{t},v(\bar{y},\bar{t}),p + q_y) \leq 0. \tag{2.18}$$

We next write a string of inequalities to be used in conjunction with (2.16) and

then explain them immediately following: We have

$$
(2.19)\qquad
\begin{aligned}
&H(\bar{x},\bar{t},u(\bar{x},\bar{t}),p + q_x) - H(\bar{y},\bar{t},v(\bar{y},\bar{t}),p + q_y) > \\
&> H(\bar{x},\bar{t},v(\bar{y},\bar{t}),p + q_x) - H(\bar{y},\bar{t},v(\bar{y},\bar{t}),p + q_y) > \\
&> H(\bar{x},\bar{t},v(\bar{y},\bar{t}),p) - H(\bar{y},\bar{t},v(\bar{y},\bar{t}),p) - (m_{R_\varepsilon}(|q_x|) + m_{R_\varepsilon}(|q_y|)) > \\
&> -(m_H(|\bar{x} - \bar{y}||p| + |\bar{x} - \bar{y}|) + m_{R_\varepsilon}(|q_x|) + m_{R_\varepsilon}(|q_y|)).
\end{aligned}
$$

The first inequality is due to the monotonicity (H2) of H and $u(\bar{x},\bar{t}) > v(\bar{y},\bar{t})$. The second inequality arises from (H1) when R_ε denotes a bound on $|p|$, $|p + q_x|$, and $|p + q_y|$. It is clear from (2.16) and (2.17) that

$$
(2.20)\qquad (\sqrt{\varepsilon})|p| < C \text{ and } |q_x|,\ |q_y| < C\sqrt{\beta}
$$

for a suitable constant C as ε and β tend to 0. Thus for some C we have

$$
(2.21)\qquad R_\varepsilon < C/\sqrt{\varepsilon}.
$$

The final inequality of (2.19) arises from (H3) and $p = 2(\bar{x} - \bar{y})/\varepsilon$ from (2.17). Using (2.16) again we have $|\bar{x} - \bar{y}| \to 0$ and

$$
(2.22)\qquad |\bar{x} - \bar{y}||p| = 2|\bar{x} - \bar{y}|^2/\varepsilon < 2\delta(\beta) + m(2(m(1)\varepsilon)^{1/2}).
$$

We are now essentially done: Letting $\beta \downarrow 0$ and then $\varepsilon \downarrow 0$ in (2.19) yields 0 on the right hand extreme of (2.19) because of (2.20) and (2.22), so (2.18) implies that $c < 0$, a contradiction.

Proof of Comparison for (SP).

The proof of the comparison result for (SP) has much in common with that for (CP), and where it differs it is simpler. Assuming the hypotheses of Theorem 2 we first prove that:

Lemma 5. Let u and v be as in the comparison assertion of Theorem 2. Then $u - v$ is bounded from above.

Proof. Let m be a modulus and $r > 0$ be such that

$$
(2.23)\qquad
\begin{aligned}
&|u(x) - u(y)| + |v(x) - v(y)| < m(|x - y|) \text{ for } x,y \in \bar{\Omega} \\
&\text{such that } |x - y| < r.
\end{aligned}
$$

Analogously to (2.3) we have

$$(2.24)\qquad \begin{aligned} &u(x) - v(y) \le m(|x - y|) + m(\min(d(x),d(y)) \text{ for } x,y \in \bar{\Omega} \\ &\text{such that } |x - y| \le r \text{ and } \min(d(x),d(y)) \le r. \end{aligned}$$

Choose a point $a \in \bar{\Omega}$ such that

$$(2.25)\qquad u(a) - v(a) > m(r),$$

(if there is no such a we are done) and then $d(a) > r$ by (2.24). The function Ψ defined by

$$(2.26)\qquad \begin{aligned} &K = 1 + 2m(r)/r^2 \text{ and} \\ &\Psi(x,y) = u(x) - v(y) - K(|x - a|^2 + |y - a|^2), \end{aligned}$$

satisfies $\Psi(a,a) > \Psi(x,y)$ for $(x,y) \in \partial B_r(a,a))$ and so it will have an interior maximum $(\bar{x},\bar{y})$ with respect to this set. We then use the definition of viscosity sub - and supersolutions to find that

$$u(\bar{x}) - v(\bar{y}) + H(\bar{x},u(\bar{x}),2K(\bar{x} - a)) - H(\bar{y},v(\bar{y}),-2K(\bar{y} - a)) \le 0.$$

Since $u(\bar{x}) > v(\bar{y})$ (by (2.23) and $\Psi(\bar{x},\bar{y}) > \Psi(a,a)$) we use (H1) - (H2) as in the proof of Theorem 1 to conclude that $u(\bar{x}) - v(\bar{y})$ is bounded above and then

$$u(a) - v(a) \le u(\bar{x}) - v(\bar{y}) + 2m(r)$$

to conclude that $u - v$ is bounded above.

To complete the proof of Theorem 2, we assume that

$$(2.27)\qquad M = \sup\,(u(x) - v(x)) > 0,$$

now choose $\varepsilon > 0$ and $\beta > 0$ and let $(\bar{x}, \bar{y})$ be a maximum point of

$$\Psi(x,y) = u(x) - v(y) -(|x - y|^2/\varepsilon + \beta(|x|^2 + |y|^2))$$

over the set $S = \{(x,y) \in \bar{\Omega} \times \bar{\Omega}: |x - y| \le r\}$, which exists since

$$(2.28)\qquad M + m(|x - y|) - (|x - y|^2/\varepsilon + \beta(|x|^2 + |y|^2)) \ge \Psi(x,y)$$

on S. Moreover,

$$(2.27)\qquad \Psi(\bar{x},\bar{y}) \ge M - \delta(\beta)$$

where $\delta(0+) = 0$. As before, one rules out $\bar{x}, \bar{y} \in \partial S$ for β, ε, r small and then, using the assumptions, if $p = 2(\bar{x} - \bar{y})$, $q_x = 2\beta\bar{x}$, $q_y = -2\beta\bar{y}$ we have

$$M - \delta(\beta) < u(\bar{x}) - v(\bar{y}) < H(\bar{y},v(\bar{y}),p + q_y) - H(\bar{x},u(\bar{x}),p + q_x).$$

Finally, arguing as in the proof for (CP), $|p|$ is bounded by $C/\sqrt{\varepsilon}$ for small ε and β, while with ε fixed q_x and q_y tend to zero like $\sqrt{\beta}$ and the proof is completed just as before.

Remark 5. One knows, in dealing with bounded solutions, that uniform continuity of u or v (and not necessarily both) suffices to prove comparison results. The analogous statement in the present context is that the above proofs easily adapt to prove that the comparison assertions of Theorem 1 remain correct provided only that u and v are continuous, there is a constant K such that

$$|u(x,t) - u(y,t)| + |v(x,t) - v(y,t)| < K(1 + |x - y|)$$

for (x,t), $(y,t) \in \bar{\Omega} \times [0,T]$ and one of u,v lies in $UC_x(\bar{\Omega} \times [0,T])$. The analogous statement for (SP) arises upon letting u and v be independent of t. Later we use this remark and the existence results of Section 2 to improve the comparison results in the case $\Omega = \mathbb{R}^N$.

Section 2. Moduli of Continuity and Existence.

Throughout this section we assume that H satisfies (H0) - (H3) in the case of (CP) and, as before, interpret these conditions and others to be laid below in the case of (SP) in the obvious ways. To begin, following Ishii [8], we label the following structure properties of H: C will be a constant such that

(3.1) $$|H(x,t,0,0)| < C(1 + |x|)$$

or

(3.2) $$|H(x,t,u,0) - H(y,t,u,0)| < C(1 + |x - y|)$$

for $x,y \in \mathbb{R}^N$, $u \in \mathbb{R}$, $t \in [0,T]$. The existence of such constants C is guaranteed by the assumptions. Contrary to the previous section, we will begin with the simpler case of (SP) and prove:

Theorem 3. Let $H \in C(\mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N)$ satisfy (H0) - (H3). If also (3.2) holds, then there is a modulus m depending only on C,

(3.3) $$\sigma(R) = m_R(R),$$

(where m_R is from (H1)), and m_H (from (H3)) such that if $u \in UC(\mathbb{R}^N)$ is a viscosity solution of (SP) then

(3.4) $$|u(x) - u(y)| \leq m(|x - y|) \text{ for } x,y \in \mathbb{R}^N.$$

Moreover, if (3.2) holds, then

(3.5) $$|u(x)| \leq C(1 + |x|) + \sigma(C).$$

<u>Proof of Theorem 3.</u>

We begin with the estimate (3.5). The proof, given our comparison result, is essentially the same as the proof of [8, Proposition 2.2]. The only property of σ needed is

(3.6) $$|H(x,t,u,p) - H(x,t,u,0)| \leq \sigma(R) \text{ for } |p| \leq R.$$

In order to guarantee that

$$v(x) = A|x| + B,$$

where $A, B \geq 0$, solves $v + H(x,v,Dv) \geq 0$, we observe that

$$v + H(x,v,Dv) \geq A|x| + B + H(x,0,AD|x|) \geq$$
$$\geq A|x| + B + H(x,0,0) - \sigma(A) \geq A|x| + B - C(1 + |x|) - \sigma(A),$$

and then take $A = C$ and $B = C + \sigma(C)$. (We are being a little bit formal; however, the subdifferential of $|x|$ at the origin is the unit ball, and so $D|x|$ stands for an arbitrary vector in the ball at $x = 0$, and the computation is valid.) The estimate (3.5) with u in place of $|u|$ now follows from comparison. One then estimates u below by $-v$ in a similar way to complete the proof.

Next, we can prove in a similar way that if C is from (3.2), then

(3.7) $$|u(x) - u(y)| \leq C|x - y| + (C + 2\sigma(C)).$$

Indeed, observe that

(3.8) $$z(x,y) = u(x) - u(y)$$

is a viscosity solution of

$$z + H(x,u(x),D_x z) - H(y,u(y),-D_y z) = 0,$$

and so the monotonicity property of H guarantees that

$$z(x,y) + H(x,u(y),D_x z) - H(y,u(y),-D_y z) \le 0 \text{ on } \{z > 0\}. \tag{3.9}$$

Define $\bar{H}: R^N \times R^N \times R^N \times R^N = R^{2N} \times R^{2N} \to R$ by

$$\bar{H}(x,y,p,q) = \inf_{r \in R}(H(x,r,p) - H(y,r,-q)). \tag{3.10}$$

Clearly (3.9) and (3.10) imply

$$z + \bar{H}(x,y,D_x z,D_y z) \le 0 \text{ on } \{z > 0\}, \tag{3.11}$$

and it is also obvious that $\bar{H}$ satisfies (H0) - (H3) on $R^{2N} \times R^{2N}$. We seek solutions

$$v(x,y) = A|x - y|^{\gamma} + B \tag{3.12}$$

of

$$v + \bar{H}(x,y,D_x v,D_y v) \ge 0 \text{ on } R^{2N}, \tag{3.13}$$

where $A,B > 0$ and $0 < \gamma \le 1$. We make some formal (when $x = y$) estimates below which one can (following Ishii) make rigorous by first replacing $|x - y|^{\gamma}$ by $(|x - y|^2 + \varepsilon)^{\gamma/2})$ and then letting $\varepsilon \to 0$. We estimate for all r:

$$\begin{aligned} v + H(x,r,D_x v) - H(y,r,-D_y v) &\ge A|x - y|^{\gamma} + B + \\ H(x,r,\gamma A|x - y|^{\gamma-2}(x - y)) &- H(y,r,\gamma A|x - y|^{\gamma-2}(x - y)). \end{aligned} \tag{3.14}$$

In the case $\gamma = 1$ we use (3.2) and (3.6) to estimate the right hand side of (3.14) below by

$$A|x - y| + B - (C(1 + |x - y|) + 2\sigma(A))$$

so v satisfies (3.13) if

$$A = C \text{ and } B = 2\sigma(C) + C.$$

Since $z \le v$ on $\partial\{z > 0\}$, we use the comparison result for (SP) to conclude that (3.7) holds. We will use the bound (3.7) while estimating the modulus of continuity. Let $\gamma < 1$. We will compare z and v only on the set

$$\{z > 0\} \cap \{|x - y| < 1\}$$

so we want $z \le v$ on the boundary of this set. By (3.7) this will hold if

$$A \ge 2C + 2\sigma(C), \tag{3.15}$$

We guarantee this by setting, for reasons which will soon be evident,

$$(3.16) \qquad A = \max(2C + 2\sigma(C),\ m_H(1) + 1).$$

Next we use (H3) to estimate the right-hand side of (3.14) below by

$$A|x - y|^\gamma + B - m_H(\gamma A|x - y|^\gamma + |x - y|)$$

and, thinking of $A|x - y|^\gamma$ as r, we see this is nonnegative on $\{|x - y| < 1\}$ provided that

$$B > K(\gamma) = \max\{m_H(\gamma r + (r/A)^{1/\gamma}) - r\colon 0 < r < A\}.$$

Since $A > m_H(1)$ by (3.16), Lemma 1 yields

$$(3.17) \qquad K(0+) = 0.$$

We now have shown that

$$u(x) - u(y) < K(\gamma) + A|x - y|^\gamma \text{ for } 0 < \gamma < 1 \text{ and } |x - y| < 1,$$

where A is independent of γ and (3.17) holds. Thus

$$m(r) = \inf\{K(\gamma) + Ar^\gamma\colon 0 < \gamma < 1\}$$

provides the desired modulus on $[0,1]$.

<u>Proof of Existence for (SP)</u>. It follows from Theorem 3 and the uniqueness that if $\{H_n\}$ is a sequence of Hamiltonians on $\mathbb{R}^N\times\mathbb{R}\times\mathbb{R}^N$ satisfying (H0)-(H3), (3.1) and (3.2) uniformly (with the same m_R, m_H, C), $u_n \in UC(\mathbb{R}^N)$ are solutions of

$$u_n + H_n(x,u_n,Du_n) = 0$$

and H_n tends locally uniformly (on $\mathbb{R}^N\times\mathbb{R}\times\mathbb{R}^N$) to a limit H, then there is a solution $u \in UC(\mathbb{R}^N)$ of $u + H(x,u,Du) = 0$ and $u_n \to u$ locally uniformly. Thus existence follows if, given H satisfying the assumptions, we can produce sequences $\{H_n\}$ and $\{u_n\}$. Let $\theta \in C^\infty(\mathbb{R})$ be nonnegative, symmetric, $\theta(r) = 1$ for $|r| < 1$ and $\theta(r) = 0$ for $|r| > 2$. Put $H_n(x,u,p) = H(x,u,\theta(|p|/n)p)$. Clearly H_n satisfies the required conditions uniformly and also, for fixed n, H_n satisfies (H0) - (H2), (H3)'. Ishii's existence theory may be invoked to provide the u_n and we are done.

We turn to the Cauchy Problem.

Modulus of Continuity for (CP).

Theorem 4. Let $H \in C(\mathbb{R}^N \times [0,T] \times \mathbb{R} \times \mathbb{R}^N)$ satisfy (H0) - (H3), (3.1), (3.2) and (3.6). Let $u \in UC_x(\mathbb{R}^N \times [0,T])$ be a viscosity solution of $u_t + H(x,t,u,Du) = 0$ on $\mathbb{R}^N \times (0,T]$. Let C_0 be a constant, $\varphi(x) = u(x,0)$ and m_φ be a modulus such that

$$|\varphi(x)| \le C_0(1 + |x|) \text{ and } |\varphi(x) - \varphi(y)| \le C_0(1 + |x - y|),$$
$$\text{and } |\varphi(x) - \varphi(y)| \le m_\varphi(|x - y|) \text{ for } x,y \in \mathbb{R}^N. \tag{3.17}$$

Then there is a modulus m depending only on C (from (3.1), (3.2)), C_0, m_φ (from (3.17)), m_H (from (H3)) and T such that

$$|u(x,t) - u(y,t)| \le m(|x - y|) \text{ for } x,y \in \mathbb{R}^N \text{ and } t \in [0,T]. \tag{3.18}$$

Moreover,

$$|u(x,t)| \le (1 + t)(A_0|x| + B_0) \text{ and}$$
$$|u(x,t) - u(y,t)| \le (1 + t)(A_0|x - y| + B_0), \tag{3.19}$$

where

$$A_0 = \max(C,C_0),\ B_0 = \max(C_0,C + 2\sigma((1+T)A_0)). \tag{3.20}$$

Also for each $R > 0$ there is a modulus m^R depending only on C, C_0, m_φ, σ, m_H, T and F where

$$F(R) = \max\{|H(z,t,r,0)|: |z| \le R,\ t \in [0,T],\ |r| \le A_0R+B_0\},$$

such that

$$|u(x,t) - u(x,s)| \le m^R(|t - s|) \text{ for } x,y \in B_R(0) \text{ and } t,s \in [0,T]. \tag{3.21}$$

Proof. The strategy of proof is the same as in the previous case. To verify the first inequality of (3.19), one shows that

$$v = (1 + t)(A|x| + B)$$

solves $v_t + H(x,t,v,Dv) \ge 0$ and $v(x,0) \ge u(x,0) = \varphi(x)$ provided that

$$A|x| + B \ge \max(C_0(|x| + 1),\ C(1 + |x|) + \sigma((1 + T)A)),$$

so the values given by (3.20) work.

We next set

$$z(x,y,t) = u(x,t) - u(y,t)$$

which, by Lemma 1 and (H2), is a solution of

$$z_t + \bar{H}(x,y,D_x z,D_y z) \le 0 \text{ on } \{z > 0\} \tag{3.22}$$

where

$$\bar{H}(x,y,p,q) = \inf\{H(x,t,r,p) - H(y,t,r,-q): r \in \mathbb{R},\ t \in [0,T]\} \tag{3.23}$$

satisfies (H0) - (H3). We seek some solutions v of $v_t + \bar{H}(x,y,D_x v,D_y v) \ge 0$ on $\{|x - y| < 1\}$ on the form

$$v(x,y,t) = (1 + t)(A|x - y|^{\gamma} + B) \tag{3.24}$$

where A, B > 0 and $0 < \gamma \le 1$. If $\gamma = 1$ it suffices to have

$$A|x - y| + B \ge C(1 + |x - y|) + 2\sigma((1 + T)A),$$

since

$$C(1 + |x - y|) + 2\sigma((1 + T)A) \ge$$

$$\ge H(y,t,r,-(1 + t)AD_y|x - y|) - H(x,t,r,(1 + t)AD_x|x - y|).$$

We also have $v(x,y,0) \ge u(x,0) - u(y,0)$ if $A|x - y| + B \ge C_0(1 + |x - y|)$. By comparison we conclude that the second inequality of (3.19) holds.

In the case $\gamma < 1$, the inequation is satisfied by v on $\{|x - y| < 1\}$ if

$$A|x - y|^{\gamma} + B \ge m_H((1 + T)\gamma A|x - y|^{\gamma} + |x - y|) \tag{3.25}$$

while $v(x,y,0) \ge z(x,y,0)$ holds on $\{|x - y| < 1\}$ if

$$A|x - y|^{\gamma} + B \ge m_{\varphi}(|x - y|) \text{ on } \{|x - y| < 1\} \tag{3.26}$$

and, by (3.19), $v(x,y,t) \ge u(x,t) - u(y,t)$ on $\{|x - y| = 1\}\times[0,T]$ if

$$A + B \ge A_0 + B_0. \tag{3.27}$$

We achieve all these conditions as follows: Put

$$A = \max(A_0 + B_0,\ m_H(1) + 1,\ m_{\varphi}(1) + 1), \tag{3.28}$$

which guarantees (3.27), and then

$$B = K(\gamma) = \max\{m_H((1+T)\gamma r+(r/A)^{1/\gamma})-r)+m_{\varphi}((r/A)^{1/\gamma})-r:\ 0 \le r \le A\}. \tag{3.29}$$

by Lemma 1

$$K(0+) = 0 \tag{3.30}$$

and so the estimate

$$u(x) - u(y) \le A|x - y|^{\gamma} + K(\gamma) \text{ for } |x - y| < 1,\ 0 < \gamma \le 1, \tag{3.31}$$

provides the desired modulus m.

We turn to the estimate of the modulus in t. Here we will depart more dramatically from Ishii's analysis in order to obtain some new information even in the case he treated. (We leave it to the reader to adapt Ishii's analysis to the current case.) To understand what follows, let us consider two extremely simple cases of the problem

(CP) $\qquad u_t + H(x,t,u,Du) = 0$ on $\mathbb{R}^N\times(0,T]$ and $u(x,0) = \varphi(x)$ on $\mathbb{R}^N$.

In the first case, put $H(x,t,p) = f(x)$. The solution of (CP) is then $u(x,t) = \varphi(x) + tf(x)$. If f is unbounded, then u is unbounded and u is not uniformly continuous in t uniformly in x. In the second case, put $H(x,t,u,p) = u$. Now the solution of (CP) is $u(x,t) = e^{-t}\varphi(x)$. If φ is unbounded then u is unbounded and u is not uniformly continuous in t uniformly in x. However, these cases illustrate the only way in which unboundedness and lack of uniform continuity arise. This is the main content of the proposition we will prove next. This proposition involves the auxiliary function $U(x,t)$ which is defined by letting $t \to U(x,t)$ be the solution of the following ode intial-value problem:

(3.32) $$U' + H(x,t,U,0) = 0,\ U(x,t) = \varphi(x),$$

where ' means "d/dt". The continuity of H and the monotonicity of $H(x,t,U,0)$ in U guarantee that U is well defined on $\mathbb{R}^N\times[0,T]$ and it is easy to see that $U \in UC_x(\mathbb{R}^N\times[0,T])$. Since U is a viscosity solution of $U_t + G(x,t,U,DU) = 0$ where $G = H(x,t,U,0)$ satisfies the necessary conditions, we deduce from the above that

(3.33) $$|U(x,t)| \le A_0|x| + B_0$$

with constants from (3.20). From (3.33) and (3.32) we conclude that

(3.34) $$|U'(x,t)| \le F(R) \text{ for } |x| \le R$$

so U is Lipschitz continuous in t uniformly for bounded x. Thus the assertions concerning the modulus in t locally in x of Theorem 4 follow from the part of Theorem 4 already proved and:

Proposition 1. Let the assumptions and notation of Theorem 4 hold. Let U be defined by (3.32) where $u(x,0) = \varphi(x)$. Then there is a function $g: [0,T] \to \mathbb{R}$ with $g(0+) = 0$ and depending only on C from (3.1), (3.2), C_0, m_φ (from (3.17)), m_0 (from (H1)) and σ (from (3.3)), such that

(3.35) $$|u(x,t) - U(x,t)| < g(t) \text{ for } x \in \mathbb{R}^N,\ t \in [0,T].$$

Remark 6. The proposition suffices to establish the local modulus in t since a uniform one-sided continuity is a uniform two-sided continuity and the continuity estimate of the Proposition at $t = 0$ may be repeated at any $t > 0$.

Proof. By Lemma 2 and (H2), $z(x,y,t) = u(x,t) - U(y,t)$ is a viscosity solution of

(3.36) $$z_t + \bar{H}(x,y,D_x z,D_y z) < 0 \text{ on } \{z > 0\},$$

where

(3.37) $$\bar{H}(x,y,t,p,q) = \inf\{H(x,t,r,p) - H(y,t,r,0): r \in \mathbb{R},\ t \in [0,T]\}$$

satisfies (H0) - (H3). In the usual way, if A_0, B_0 are given by (3.20)

(3.38) $$|u(x,t) - U(y,t)| < (1 + t)(A_0|x - y| + B_0)$$

because the right-hand side is a supersolution of (3.36), etc.. Now we seek another supersolution on $|x - y| < 1$ in the form

(3.39) $$v(x,y,t) = (1 + t)A(|x - y|^2 + \varepsilon)^{\gamma/2} + tB + D$$

where A, B, $D > 0$, $0 < \gamma < 1$ and $\varepsilon > 0$, which further satisfies

(3.40) $$v(x,y,0) > m_\varphi(|x - y|) > u(x,0) - U(y,0) = \varphi(x) - \varphi(y)$$

and

(3.41) $$v(x,y,t) > (1 + T)(A_0 + B_0) > u(x,t) - U(y,t) \text{ if } |x - y| = 1.$$

If we achieve all these things, then

(3.42) $$u(x,t) - U(x,t) < v(x,x,t) < (1 + t)A\varepsilon^{\gamma/2} + tB + D$$

by comparision.

We leave it to the reader to verify that (3.41) holds for arbitrary B, $D > 0$ if

(3.43) $$A = \max((1 + T)(A_0 + B_0), m_\varphi(1) + 1),$$

and then, with A now fixed by (3.43), (3.40) holds if

(3.44) $$D = K_1(\gamma) = \max\{\, m_\varphi((r/A)^{1/\gamma})) - r\colon 0 \leq r \leq A\}.$$

Finally, we use that $|D_x A(|x - y|^2 + \varepsilon)^{\gamma/2}| \leq A\gamma/(\varepsilon^{(1-\gamma)/2})$ to conclude that v is a supersolution on $|x - y| < 1$ provided

(3.45) $$B = K_2(\gamma,\varepsilon) = \sigma(A\gamma/(\varepsilon^{(1-\gamma)/2})) + m_\varphi(1).$$

But now we are done: A is fixed, D tends to zero with γ and so the right hand side of (3.42) can be made as small as desired by taking ε small, then γ small and, finally, t small. A lower bound is produced in the same way.

Remark 7. Using a similar (but somewhat more complex) argument one can prove u $- U \in BUC(\mathbb{R}^N\times[0,T])$. In particular, if $H(x,t,u,p)$ is independent of u, the above proof adapts to show that

$$u(x,t) - \varphi(x) + \int_0^t H(x,\tau,0)d\tau$$

is uniformly continuous in t with a modulus depending only on the usual data. The analogue for (SP) of these results is that if $U(x)$ is defined implicitly by the equation

$$U(x) + H(x,U(x),0) = 0,$$

and u is the solution of (SP), then $u - U \in BUC(\mathbb{R}^N)$.

Existence for (CP).

Theorem 4 and the same device which was used to establish existence for (SP) succeed here, and we need not repeat the argument.

Remark 8. The existence results cn be used together with Remark 5 to strengthen the comparison results when $\Omega = \mathbb{R}^N$. We illustrate this for (SP) - the analogous remarks hold for (CP). Let $H \in C(\mathbb{R}^N\times\mathbb{R}\times\mathbb{R}^N)$ satisfy (H0) - (H3). Let $u,v \in C(\mathbb{R}^N)$ be a sub and a supersolution of (SP) on $\mathbb{R}^N$ such that there is a constant K for which

$$|u(x) - u(y)| + |v(x) - v(y)| \leq K(1 + |x - y|) \text{ for } x,y \in \mathbb{R}^N.$$

Then $u \leq v$. Indeed, let $w \in UC(\mathbb{R}^N)$ be the solution of (SP) provided by

Theorem 2. By Remark 5, $u \le w$ and $w \le v$, so the assertion is correct.

Remark 9. The conditions (H3) and (H3)' are not "natural" in the sense that they are not invariant under simple changes of variables. Hence, for example, they are not sensible on manifolds. To extend the scope of our results to include manifolds, we will formulate a generalization of (H3), and to this end let us consider the following properties of a metric $(x,y) \to d(x,y)$ on $\mathbb{R}^N$:

(d1) There are constants λ, Λ such that

$$\lambda|x - y| \le d(x,y) \le \Lambda|x - y| \text{ for } x,y \in \mathbb{R}^N.$$

(d2) For $z,y \in \mathbb{R}^N$ and $z \neq y$, the map $x \to d(x,y)$ is differentiable at z.

We will denote the derivative of the map $x \to d(x,y)$ at x and the map $y \to d(x,y)$ at y by $d_x(x,y)$ and $d_y(x,y)$ respectively. The generalization of (H3) is:

(H3)" There is a distance d satisfying (d1) and (d2) and a modulus m_H such that for all $(t,r) \in [0,T]\times\mathbb{R}$, $x,y \in \mathbb{R}^N$ and $\xi \ge 0$

$$H(x,t,r,\xi d_x(x,y)) - H(y,t,r,-\xi d_y(x,y)) \le -m_H(\xi|x - y| + |x - y|).$$

The uniqueness arguments go through if (H3) is replaced by (H3)" when one replaces $|x - y|$ by $d(x,y)$ at appropriate points in the proofs. Moreover, one can work more generally on manifolds. The extension of the existence assertions to manifolds deserves more than a remark and will be considered elsewhere.

BIBLIOGRAPHY

[1] Barles, G., Contrôle impulsionnel déterministe, inequations quasi-variationelles et équations de Hamilton-Jacobi du premier ordre, These de 3^{e} cycle, Université de Paris IX - Dauphine, 1983.

[2] Crandall, M. G., L. C. Evans and P. L. Lions, Some properties of viscosity solutions of Hamilton-Jacobi equations, Trans. Amer. Math. Soc. 282 (1984), 487 - 502.

[3] Crandall, M. G., and P. L. Lions, Viscosity solutions of Hamilton - Jacobi equations, Trans. Amer. Math. Soc. 277 (1983), 1 - 42.

[4] Crandall, M. G. and P. L. Lions, Solutions de viscosité non bornées des équations de Hamilton-Jacobi du premier ordre, C. R. Acad. Sci. Paris 298 (1984), 217 - 220.

[5] Crandall, M. G. and P. L. Lions, in preparation.

[6] Crandall, M. G. and R. Newcomb, Viscosity solutions of Hamilton - Jacobi equations at the boundary, Mathematics Research Center TSR # 2723, University of Wisconsin - Madison, 1984 (and to appear in Proc. Amer. Math. Soc.).

[7] Crandall, M. G., and P. E. Souganidis, Developments in the theory of nonlinear first order partial differential equations, Proceedings of the International Symposium on Differential Equations, Birmingham, Alabama (1983), North-Holland Mathematics Studies 92, North-Holland, Amsterdam, 1984.

[8] Ishii, H., Remarks on the Existence of Viscosity Solutions of Hamilton-Jacobi Equations, Bull. Facul. Sci. Eng., Chuo University, 26 (1983), 5 - 24.

[9] Ishii, H., Uniqueness of unbounded solutions of Hamilton-Jacobi equations, Indiana Univ. Math. J., to appear.

[10] Lions, P. L., Generalized Solutions of Hamilton - Jacobi Equations, Research Notes in Mathematics 69, Pitman, Boston, 1982.

[11] Lions, P. L., Existence results for first-order Hamilton-Jacobi equations, Richerche Mat. Napoli, 32 (1983), 1 - 23.

[12] Souganidis, P. E., Existence of viscosity solutions of Hamilton - Jacobi equations, J. Diff. Eq., to appear.

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The theory of scalar first order nonlinear partial differential equations has been enjoying a rapid development in the last few years. This development occurred because the authors established uniqueness criteria for generalized solutions - called viscosity solutions - which correctly indentify the solutions sought in areas of applications, including control theory, differential games and the calculus of variations. The concept of viscosity solutions is relatively easy to work with and many formally heuristic or difficult proofs have been made rigorous or simple using this concept. A feedback process has begun and

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ABSTRACT (continued)

the experience recently gained in working with viscosity solutions has suggested new existence and uniqueness results. The current paper continues this interaction by establishing new existence and uniqueness results in a natural generality suggested by earlier proofs. It is also felt that the presentation of the comparison results, which imply uniqueness, continuous dependence, and are used to estimate moduli of continuity, has something to offer over earlier presentations in special cases.